MATHEMATICS

ON THE STRUCTURE OF WELL DISTRIBUTED SEQUENCES (III)

BY

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1. Firstly, we shall discuss the relation of Tauberian theorems to the distribution of sequences. We begin with some definitions and notation.

Let (s_n) be a sequence of real numbers satisfying $0 \leqslant s_n \leqslant 1$ for every n, $n=1, 2, 3, \ldots$. We take $0 \leqslant a < b \leqslant 1$ and let $I_{[a,b]}(x)$ denote the characteristic function of the interval $[a, b]$, so that

$$I_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

The sequence (s_n) is said to be *well distributed* [1]) if

$$\lim_{p\to\infty} \frac{1}{p} \cdot \sum_{k=n+1}^{n+p} I_{[a,b]}(s_k) = b-a$$

holds uniformly in n for every interval $[a, b]$. In other words, (s_n) is well distributed if the sequence $(I_{[a,b]}(s_n))$ is *almost convergent* [2]) to $b-a$ for every subinterval $[a, b]$ of the interval $[0, 1]$.

Consider the series $\sum\limits_{k=1}^{\infty} \omega_k$ with partial sums

$$s_n = \sum_{k=1}^{n} \omega_k,$$

and let $A=(a_{mn})$ be a regular summability matrix (or the method of almost convergence $\mathscr{A}$). If A (or $\mathscr{A}$)-summability of the sequence (s_n), together with the condition

$$(1) \qquad |\omega_k| = 0(|f(k)|), \text{ where } \lim_{k\to\infty} f(k) = 0$$

implies that the series $\sum\limits_{k=1}^{\infty} \omega_k$ is convergent in the ordinary sense, then (1) is said to be a *Tauberian condition* for A (or $\mathscr{A}$).

The sequence (s_n) is said to be *A-uniformly distributed* [3]) if, for each continuous function $g(x)$ with period 1, the sequence $(g(s_n))$ is A-summable to the value $\int_0^1 g(x)\,dx$, that is, for each such function $g(x)$,

$$(2) \qquad \lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk}\, g(s_k) = \int_0^1 g(x)\,dx.$$

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[1]) See [1], § 2 page 189.
[2]) See [2], pages 169, 170.
[3]) See [3], page 45.

Notation

Throughout, we shall use $\{\theta\}$ to denote the *fractional part* of θ, that is $\theta-[\theta]$, where $[\theta]$ is the largest integer less than or equal to θ.

We are now in a position to state the following:

Theorem 1. *If the sequence* $(\omega_n)=(s_n-s_{n-1})$ *satisfies a Tauberian condition for* A (*or* A), *as in* (1), *then the sequence* $(\{s_n\alpha\})$ *is not* A*-uniformly distributed* (*well distributed*) *for any* α, $0<\alpha\leqslant 1$.

Proof. Let g be a function with period 1 which is defined on the interval $[0, 1]$ as follows:

$$g : g(x) = \begin{cases} x & \text{for } 0\leqslant x<\frac{1}{2} \\ 1-x & \text{for } \frac{1}{2}\leqslant x<1. \end{cases}$$

Since the sequence $(\omega_n)=(s_n-s_{n-1})$ satisfies a Tauberian condition similar to (1), there exists a positive integer $N=N(\alpha)$ such that, for $n>N(\alpha)$, we have

$$|s_n\alpha-s_{n-1}\alpha| < \tfrac{1}{2},$$

where α is a given real number with $0<\alpha\leqslant 1$. There are now a number of cases to be considered.

(i) Suppose that $0\leqslant\{s_n\alpha\}<\frac{1}{2}$ and $0\leqslant\{s_{n-1}\alpha\}<\frac{1}{2}$, then

$$\begin{aligned} |g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})| &= |\{s_n\alpha\}-\{s_{n-1}\alpha\}| \\ &= |s_n\alpha-s_{n-1}\alpha| , \end{aligned}$$

since, in this case, $[s_n\alpha]=[s_{n-1}\alpha]$. Thus

$$\begin{aligned} |g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})| &= \alpha|s_n-s_{n-1}| \\ &= 0(|f(n)|). \end{aligned}$$

(ii) Likewise, if $\frac{1}{2}\leqslant\{s_n\alpha\}<1$ and $\frac{1}{2}\leqslant\{s_{n-1}\alpha\}<1$, then $[s_n\alpha]=[s_{n-1}\alpha]$, and so

$$\begin{aligned} |g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})| &= |\{s_n\alpha\}-\{s_{n-1}\alpha\})| \\ &= |s_n\alpha-s_{n-1}\alpha| \\ &= 0(|f(n)|). \end{aligned}$$

(iii) If $\frac{1}{2}\leqslant\{s_n\alpha\}<1$, $0\leqslant\{s_{n-1}\alpha\}<\frac{1}{2}$, and $[s_n\alpha]=[s_{n-1}\alpha]$, then

$$\begin{aligned} |g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})| &= |1-\{s_n\alpha\}-\{s_{n-1}\alpha\}| \\ &= |\tfrac{1}{2}-\{s_n\alpha\}+(\tfrac{1}{2}-\{s_{n-1}\alpha\})| \\ &\leqslant |\tfrac{1}{2}-\{s_n\alpha\}|+|\tfrac{1}{2}-\{s_{n-1}\alpha\}| \\ &= \{s_n\alpha\}-\tfrac{1}{2}+\tfrac{1}{2}-\{s_{n-1}\alpha\} \\ &= \{s_n\alpha\}-\{s_{n-1}\alpha\} \\ &= s_n\alpha-s_{n-1}\alpha, \end{aligned}$$

since $[s_n\alpha]=[s_{n-1}\alpha]$. Thus

$$|g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})|=s_n\alpha-s_{n-1}\alpha$$
$$=0(|f(n)|).$$

Similarly, if $\frac{1}{2}\leqslant\{s_{n-1}\alpha\}<1$, $0\leqslant\{s_n\alpha\}<\frac{1}{2}$, and $[s_n\alpha]=[s_{n-1}\alpha]$, we get

$$|g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})|=s_{n-1}\alpha-s_n\alpha$$
$$=0(|f(n)|).$$

(iv) Suppose now that $\frac{1}{2}\leqslant\{s_n\alpha\}<1$, $0\leqslant\{s_{n-1}\alpha\}<\frac{1}{2}$, and $[s_n\alpha]\neq[s_{n-1}\alpha]$. We recall that, for $n>N(\alpha)$,

$$|s_n\alpha-s_{n-1}\alpha|<\tfrac{1}{2},$$

and hence

$$[s_{n-1}\alpha]=[s_n\alpha]+1,$$

for $[s_n\alpha]=[s_{n-1}\alpha]+1$ would imply

$$|s_n\alpha-s_{n-1}\alpha|>\tfrac{1}{2}.$$

Hence, for $n>N(\alpha)$,

$$\begin{aligned}|g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})|&=|1-\{s_n\alpha\}-\{s_{n-1}\alpha\}|\\&\leqslant|1-\{s_n\alpha\}|+|\{s_{n-1}\alpha\}|\\&=1-\{s_n\alpha\}+\{s_{n-1}\alpha\}\\&=1-(s_n\alpha-[s_n\alpha])+s_{n-1}\alpha-[s_{n-1}\alpha]\\&=1-s_n\alpha+[s_n\alpha]+s_{n-1}\alpha-[s_n\alpha]-1\\&=s_{n-1}\alpha-s_n\alpha=0(|f(n)|).\end{aligned}$$

Similarly, if $\frac{1}{2}\leqslant\{s_{n-1}\alpha\}<1$, $0\leqslant\{s_n\alpha\}<\frac{1}{2}$, and $[s_n\alpha]\neq[s_{n-1}\alpha]$, then, for $n>N(\alpha)$, we obtain

$$|g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})|\leqslant s_n\alpha-s_{n-1}\alpha=0(|f(n)|).$$

All possible cases have now been investigated and for each we have

$$|g(\{s_n\alpha\})-g(\{s_{n-1}\alpha\})|\leqslant 0(|f(n)|),$$

where

$$\lim_{n\to\infty} f(n)=0,$$

and hence the sequence $(g(\{s_n\alpha\}))$ satisfies a Tauberian condition of the form (1). Furthermore, if equation (2) is also satisfied, then $(g(\{s_n\alpha\}))$ must be a convergent sequence. In addition, $(\{s_n\alpha\})$ is either not A-uniformly distributed (well distributed) or else is everywhere dense in [0, 1]. In this latter case, the number 0 is an accumulation point of the sequence $(\{s_n\alpha\})$, and $(g(\{s_n\alpha\}))$ converges to zero and is A-summable (almost convergent) to zero. However,

$$\int_0^1 g(x)\,dx=\frac{1}{4},$$

and it is clear that $(\{s_n\alpha\})$ is not A-uniformly distributed (well distributed) for any α.

For almost convergence $\mathscr{A}$, any function $f(k)$ satisfying (1) is a Tauberian condition. This implies that the following theorem is true, and this may be compared with Theorem A of [4].

Theorem 2. *If the sequence* $(\omega_n)=(s_n-s_{n-1})$ *satisfies*

$$\omega_n=o\ (1),$$

then $\{s_n\alpha\}$ *is not well distributed for any* α.

By a closer examination of the proof of Theorem 1, we could in fact modify the remark made at the end of [4] to read: "If $\{f(k)\alpha\}$, $0<\alpha\leqslant\leqslant 1$, $f(k)\nearrow\infty$, is well distributed for any α, then

$$f(k)>k\cdot r'$$

for some r' and $k=K, K+1, \ldots$."

2. An examination of the various theorems contained in [5] leads us to present the following conjecture: "*If* $(n(k))$ *is a sequence of real numbers such that*

$$\frac{n(k)}{n(k-1)}\geqslant r>1, \qquad (k=2, 3, \ldots)$$

then $\{n(k)\alpha\}$ *is not well distributed for almost all* α." Clearly, the different theorems in [5] and [6] in effect make use of the Tauberian theorems for almost convergence. Before making a contribution to resolving the above conjecture we shall present some definitions and state Theorem F of [5].

Let $(n(k))$ be a subsequence of the integers. For the subsequence $(n(k_i))$ of the sequence $(n(k))$ we define the *index sequence* (x_j) of $(n(k_i))$ by the equations

$$x_j=\begin{cases}1 \text{ when } j=k_1, k_2, \ldots\\ 0 \text{ otherwise.}\end{cases}$$

Let us suppose that, for some positive integer R,

$$\frac{1}{p}\cdot\sum_{j=m+1}^{m+p} x_j>\frac{1}{R}>0$$

for all m, and all $p>P$. If this is so we say that the *lower density* of $(n(k_i))$ exceeds $1/R$, or $(n(k_i))$ has a lower density that is positive.

Theorem F. *If* $(n(k))$ *possesses a subsequence* $(n(k_i))$ *satisfying*

(3) $$\lim_{i\to\infty}\frac{n(k_i)}{n(k_{i-1})}=\infty,$$

and having a lower density that is positive, then $\{n(k)\alpha\}$ *is not well distributed for almost all* α, $0<\alpha<1$.

We are now in a position to prove the following:

Theorem 3. *Let $(n(k))$ be a subsequence of the integers such that*

$$\frac{n(k)}{n(k-1)} \text{ is an integer,} \qquad (k=2, 3, \ldots) \tag{4}$$

and

$$\frac{n(k)}{n(k-1)} \geqslant 2, \qquad (k=2, 3, \ldots) \tag{5}$$

then, for almost all α, $0<\alpha\leqslant 1$, the sequence $\{n(k)\alpha\}$ is not well distributed.

Proof. Let us denote the ratio $n(k)/n(k-1)$ by $r(k)$, $(k=2, 3, \ldots)$, and set $n(1)=r(1)$. Then the real number α has a unique expansion [1] of the form

$$\alpha = \sum_{k=1}^{\infty} \frac{a(k)}{r(1) \ldots r(k)} = \sum_{k=1}^{\infty} \frac{a(k)}{n(k)},$$

where $a(k)$, $(k=1, 2, \ldots)$, is an integer satisfying the further conditions

$$0 \leqslant a(k) \leqslant r(k)-1.$$

Using (4) and (5) it is evident that

$$\{n(k)\alpha\} = \sum_{\nu=1}^{\infty} \frac{a(k+\nu)}{r(k+1) \ldots r(k+\nu)},$$

and further, this number will lie in the interval $[0, \frac{1}{2}]$ if $a_{k+1}=0$. Hence, if we can show that, for almost all α, there is a sequence of indices (k_r) (this sequence will, in general, be dependent upon α) such that

$$a(k_r+1)=a(k_r+2) = \ldots = a(k_r+r)=0, \qquad (r=1, 2, \ldots)$$

then it is clear that

$$\{n(k)\alpha\} \leqslant \tfrac{1}{2}, \qquad (k=k_r, k_r+1, \ldots, k_r+r-1;\ r=1, 2, \ldots).$$

This implies that

$$\frac{1}{r} \sum_{k=k_r}^{k_r+r-1} I_{[0,\frac{1}{2}]}(\{n(k)\alpha\}) = 1 \tag{6}$$

for infinitely many r and, in view of the definition of a well distributed sequence, $\{n(k)\alpha\}$ is not well distributed for almost all α.

In general it would not be possible to establish the existence of such a sequence of indices for almost all α. However, if every subsequence $(n(k_i))$ of $(n(k))$ satisfying (3) [2] has a lower density that is not positive, i.e. is zero, then we shall show that such a sequence of indices (k_r) exists

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1) See, for example, [7].

2) Note that if (5) is satisfied then there exists a subsequence $(n(k_i))$ such that (3) is also satisfied.

for almost all α. This will complete the proof of Theorem 3 since Theorem F covers the possibility of (3) being satisfied for the subsequence $(n(k_i))$ with positive lower density.

We therefore assume that any subsequence $(n(k_i))$ satisfying (3) has lower density zero. Let E be the set of all α's for which there exists a sequence of indices (k_r) such that

$$a(k)=0, \qquad (k=k_r+1, \ldots, k_r+r;\ r=1, 2, \ldots).$$

Then, if E_ν is the set of α's for which there exists a sequence of indices (k_t) such that

$$a(k)=0, \qquad (k=k_t+1, \ldots, k_t+\nu;\ t=1, 2, \ldots)$$

where ν is fixed, it is clear that every α belonging to E belongs to E_ν, $(\nu=1, 2, \ldots)$, and hence

$$E \subseteq \bigcap_{\nu=1}^{\infty} E_\nu,$$

and, in fact, that

$$E = \bigcap_{\nu=1}^{\infty} E_\nu.$$

Consequently, if

$$\mu(E_\nu)=1, \qquad (\nu=2, 3, \ldots)$$

then $\mu(E)=1$, and our theorem is proved.

Now $\mu(E_\nu)=1$ if $\mu(E_\nu')=0$, where E_ν' is the complement of E_ν. We have that

$$E_\nu' \subseteq E_\nu^*,$$

where E_ν^* is the set of α's for which

$$a(\nu k-\nu+1)=a(\nu k-\nu+2) = \ldots = a(\nu k)=0 \tag{7}$$

for at most a finite set of k's. Some points of E_ν^* may in fact belong to E_ν, but every point of E_ν' will be in E_ν^*. Finally,

$$E_\nu^* = \bigcup_{p=1}^{\infty} {}_pE_\nu^*,$$

where ${}_pE_\nu^*$, $(p=1, 2, \ldots)$, is the set of α's for which (7) is not satisfied for any $k \geqslant p$. We shall show that, for each ν,

$$\mu({}_pE_\nu^*)=0, \qquad (p=1, 2, \ldots).$$

This implies that

$$\mu(E_\nu^*)=\mu(E_\nu')=0, \qquad (\nu=1, 2, \ldots)$$

and that $\mu(E)=1$.

The set ${}_{k_0}F_\nu$ of α's for which (7) is not satisfied for $k=k_0$ is made up of intervals of the form

$$\left(\frac{q}{n(\nu k_0)}, \frac{q+1}{n(\nu k_0)}\right), \left(q \neq m\,\frac{n(k_0\nu)}{n(k_0\nu-\nu)}, \ m = 0, 1, \ldots, n(k_0\nu-\nu)-1\right)$$

and, for $k_0=1$, $q\neq 0$. This means that there are $n(k_0\nu)-n(k_0\nu-\nu)$ intervals of length $1/n(k_0\nu)$, and clearly

$$\mu({}_{k_0}F_\nu) = 1 - \frac{n(k_0\nu - \nu)}{n(k_0\nu)}.$$

Also

$$\mu({}_pE_\nu{}^*) = \prod_{k=p}^{\infty}\left(1 - \frac{n(k\nu-\nu)}{n(k\nu)}\right).$$

This infinite product will diverge to zero if the series

$$\sum_{k=p}^{\infty} \log\left(1 - \frac{n(k\nu-\nu)}{n(k\nu)}\right)$$

diverges, and, since $|\log(1-x)| \geqslant x$ for $0 \leqslant x < 1$, this series diverges if

$$\sum_{k=p}^{\infty} \frac{n(k\nu-\nu)}{n(k\nu)} = \sum_{k=p}^{\infty} \frac{1}{r(k\nu-\nu+1)\ \dots\ r(k\nu)} \tag{8}$$

diverges. However, for (8) to converge we must have

$$\lim_{k\to\infty} r(h_k) = \infty,$$

where $h_k = k\nu - \nu + 1, \ldots, k\nu$, and since $h_{k+1} - h_k \leqslant 2\nu$, the sequence (h_k) has positive lower density. This is contrary to our assumption that (3) is satisfied only on sets with lower density zero. Our assumption therefore means that (8) diverges and that

$$\mu(E_\nu{}') = 0, \qquad (\nu = 1, 2, 3, \ldots).$$

Hence, $\mu(E) = 1$, which completes the proof of the theorem.

It is evident that (4) makes it possible to express the real numbers α in such a form that they may be subjected to analysis. However, it is remarkable that the technique used in treating the cases where (3) is satisfied only on sets of lower density zero, breaks down when this restriction is removed. Further, the techniques used in proving Theorem F of [5] cannot be extended to establish Theorem 3.

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